Mathematical Physics 111

Problem Set 4

National Institute of Physics

(Dated: November 9, 2017)

Deadline: 21 November 2017

Convention: In spherical coordinates use $q_2 = \theta \in [0, \pi]$ as the polar angle and $q_3 = \varphi \in [0, 2\pi)$ as the equatorial angle. Different conventions were inadvertently used in the lecture and main reference text.

I. SPHERICAL COORDINATES [30 pts]

Use the convention given above, which is the same used in the textbook.

10 pts Calculate all of the elements of the metric tensor $\mathbf{g}$ and arrange them in matrix form. Write down $d\mathbf{r}$ in terms of dr, $d\theta$, $d\varphi$.

10 pts Let $\mathbf{E}(r)$ be the electric field due to a stationary point charge q located at the origin. Use the spherical form of the divergence operator to calculate $\boldsymbol{\nabla} \cdot \mathbf{E}$ for $r \neq 0$.

10 pts Prove that $\boldsymbol{\nabla}^2 \psi(r) = \frac{2}{r}\frac{d\psi}{dr} + \frac{d^2\psi}{dr^2}$ for some scalar function $\psi(r)$ that depends only on the radial coordinate r.

II. ACCELERATION [40 pts]

Let the position of a particle be $\mathbf{r}(t)$ at time t. In Cartesian coordinates the acceleration of the particle is $\ddot{\mathbf{r}}(t) = \ddot{x}(t)\hat{\mathbf{x}} + \ddot{y}(t)\hat{\mathbf{y}} + \ddot{z}(t)\hat{\mathbf{z}}$.

20 pts Calculate the acceleration of the object $\ddot{\mathbf{r}}(t)$ in cylindrical coordinates.

20 pts Calculate the acceleration of the object $\ddot{\mathbf{r}}(t)$ in spherical coordinates.

III. FIELD OF A MAGNETIC DIPOLE [20 pts]

[AR5 2.5.22] Let a point magnetic dipole have dipole moment $\mathbf{m} = m\hat{\mathbf{z}}$. In coordinate free form, the vector potential for the induction field $\mathbf{B}$ due to this dipole is

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{1}$$

Derive the induction field $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$.

IV. ANGULAR MOMENTUM [20 pts]

[AR5 2.5.16] Let $\mathbf{L}\psi(\mathbf{r}) = -i(\mathbf{r} \times \boldsymbol{\nabla})\psi(\mathbf{r})$. Derive the form of the differential operator $\mathbf{L}^2$ in spherical coordinates.