Quantum Physics 142 and 204.6

Problem Set 1

National Institute of Physics

Deadline: 16 October 2020

Conventions: Let $|j, m_j\rangle$ be the simultaneous eigenket of $\mathbf{J}^2$ and J_z with corresponding eigenvalues $j(j+1)\hbar^2$ and $m_j\hbar$, respectively. Consider the angular momentum addition $\mathbf{J}_1 + \mathbf{J}_2 = \mathbf{J}$. When the quantum numbers j_1 and j_2 are fixed, we can use the following abbreviations for the decoupled basis elements:

$$|j_1, m_1\rangle\,|j_2, m_2\rangle \equiv |m_1 m_2\rangle\,, \tag{1}$$

and for the coupled basis elements:

$$|j_1\,j_2\,j, m\rangle \equiv |j, m\rangle\,. \tag{2}$$

Note the location of the comma.

I. HYDROGEN ATOM EIGENFUNCTIONS [30 pts]

Consider the Coulomb potential

$$V(\mathbf{r}) = -\frac{k}{r}, \quad k = \frac{e^2}{4\pi\epsilon_0}. \tag{3}$$

Do not consider spin.

10 pts Obtain the ground state energy eigenvalue and the corresponding eigenfunction. Write the eigenfunction in terms of the Bohr radius a_0.

20 pts Obtain the first excited state energy eigenvalue and the corresponding eigenfunction/s. If the eigenvalue is degenerate, orthonormalize the eigenfunctions.

II. HYDROGEN ATOM EIGENFUNCTIONS [30 pts]

Consider the Coulomb potential

$$V(\mathbf{r}) = -\frac{k}{r}, \quad k = \frac{e^2}{4\pi\epsilon_0}. \tag{4}$$

Do not consider spin.

10 pts Obtain the ground state energy eigenvalue and the corresponding eigenfunction. Write the eigenfunction in terms of the Bohr radius a_0.

20 pts Obtain the first excited state energy eigenvalue and the corresponding eigenfunction/s. If the eigenvalue is degenerate, orthonormalize the eigenfunctions.

III. ANGULAR MOMENTUM ADDITION [30 pts]

Let $\mathbf{J}^2 = (\mathbf{L} + \mathbf{S})^2$ and consider the angular momentum subspace with quantum numbers $l = 1$ and $s = 1/2$.

10 pts Give all of the normalized simultaneous eigenvectors of $\mathbf{L}^2, \mathbf{S}^2, \mathbf{J}^2, J_z$ in the coupled basis $\{|j, m\rangle\}$.

10 pts Use the ladder operator method or a table of Clebsch-Gordan coefficients to write each of the coupled basis elements as a linear combination of decoupled basis elements $|m_l m_s\rangle$. You may use $m_l \in \{-1, 0, +1\}$ and $m_s \in \{\uparrow, \downarrow\}$.

10 pts Give all possible outcomes of measurements of the observable $\mathbf{L}\cdot\mathbf{S}$.

IV. SPINOR [30 pts]

Given a magnetic field $\mathbf{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}$:

10 pts Write down the Hamiltonian $H = -\gamma\mathbf{B}\cdot\mathbf{S}$ as a 2×2 matrix.

20 pts Obtain the eigenspinors of this Hamiltonian and their corresponding eigenvalues.

V. SPIN PRECESSION [50 pts]

Let a spin-1/2 system be in the initial state

$$\chi(0) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}. \tag{5}$$

10 pts Determine if $\chi(0)$ is a stationary state of S_x or not.

20 pts Obtain the time development of $\chi(t)$ of this state under the Hamiltonian $\hat{H} = -\gamma B S_z$.

20 pts Show that the time development of the expectation values $\langle S_x(t)\rangle$, $\langle S_y(t)\rangle$, and $\langle S_z(t)\rangle$ describes a precessing vector. What is the frequency of precession?

VI. UNCERTAINTY RELATION [30 pts]

10 pts Use the commutation relation between angular momentum components to obtain the following uncertainty relation: $\Delta J_x \Delta J_y \geq \frac{\hbar}{2}|\langle J_z\rangle|$.

20 pts Verify this uncertainty relation for the preceding example of spin precession. That is, evaluate the needed expectation values with respect to $\chi(t)$.